Chapter 4. Linear D.E for higher order

4.1 General Theory.

IVP.

$y'' - y = 0 \qquad y = c_1 e^{x} + c_2 e^{-x}$ on $(-\infty, \infty)$

$y(0) = 0 \qquad y'(0) = 1.$

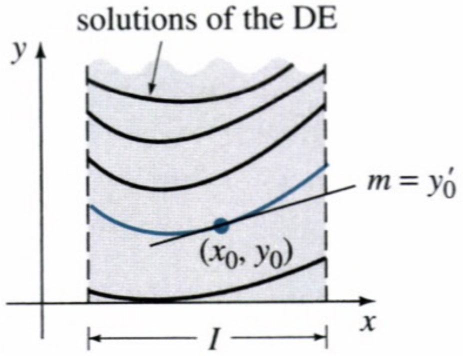


Figure 4.1

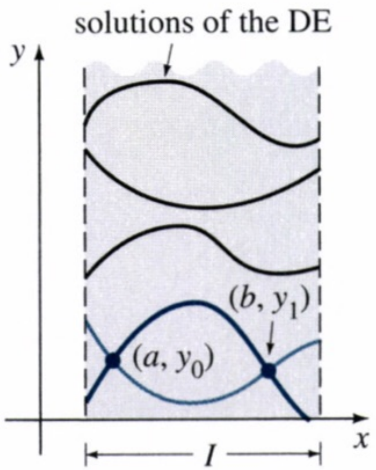
$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= c_1 e^{0} - c_2 e^{-0} = 1 \end{aligned}\right\} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 1 \end{cases} \Rightarrow 2c_1 = 1$$

$$\Rightarrow c_1 = \frac{1}{2} \Rightarrow c_2 = -\frac{1}{2} \Rightarrow y = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$$

BVP (Boundary-Value Problem)

$y'' - y = 0 \qquad y = c_1 e^{x} + c_2 e^{-x}$ on $(-\infty, \infty)$

Boundary condition $y(0) = 0 \quad y(1) = 1.$



$$y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$y(1) = c_1 e + c_2 e^{-1} = 1 \Rightarrow -c_2 e + c_2 e^{-1} = 1 \Rightarrow c_2 = \frac{1}{e^{-1} - e}$$

$$\Rightarrow c_1 = \frac{1}{e - e^{-1}} \Rightarrow y = \frac{1}{e - e^{-1}} e^{x} + \frac{1}{e^{-1} - e} e^{-x}$$

Example $y'' + 16y = 0$ $\quad y = C_1 \cos 4x + C_2 \sin 4x$

$\underbrace{y(0) = 0}_{①}$ $\quad \underbrace{y\left(\frac{\pi}{2}\right) = 0}_{②}$

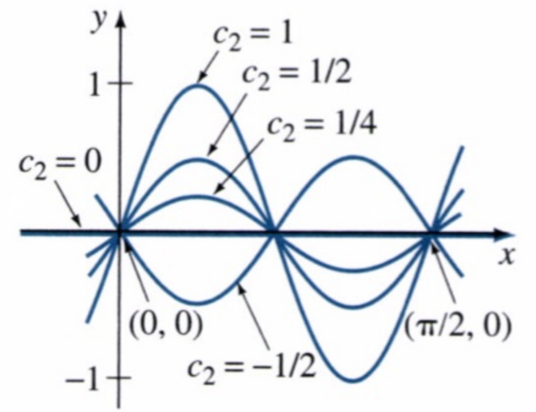
① $0 = C_1 \cos 0 + C_2 \sin 0 \Rightarrow 0 = C_1 \Rightarrow y = C_2 \sin 4x$

② $0 = C_2 \sin 4\left(\frac{\pi}{2}\right) \Rightarrow 0 = C_2 \sin 2\pi$

$x = \frac{\pi}{2}$

always true for $\forall C_2$

$\therefore$ BVP solution is $\boxed{y = C_2 \sin 4x.}$



4.1.3 Solutions of Linear Equations

Homogeneous Equations A linear nth-order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (3)$$

is said to be **homogeneous,** whereas

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = \underline{g(x)}, \qquad (4)$$

$g(x)$ not identically zero, is said to be **nonhomogeneous.**

The word *homogeneous* in this context does not refer to coefficients that are homogeneous functions. See Section 2.3.

THEOREM 4.3 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous linear nth-order differential equation (3) on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x), \qquad (5)$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

How do we describe all of them?

THEOREM 4.2 **Criterion for Linearly Independent Functions**

Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least $n-1$ derivatives. If the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is not zero for at least one point in the interval I, then the set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is linearly independent on the interval.

THEOREM 4.4 **Criterion for Linearly Independent Solutions**

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (3) on an interval I. Then the set of solutions is **linearly independent** on I if and only if

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

for every x in the interval.

DEFINITION 4.1 **Linear Dependence**

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval.

DEFINITION 4.2 **Linear Independence**

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly independent** on an interval I if it is not linearly dependent on the interval.

$x,\ 5x+2,\ 1$ are linearly dependent $\because\ 5x + 2\cdot 1 = 5x+2$

$x,\ 5x+2,$ are linearly independent $\because$ no way $5x+2 = cx$ where c is real number

DEFINITION 4.3 Fundamental Set of Solutions

Any linearly independent set $y_1, y_2, \ldots, y_n$ of n solutions of the homogeneous linear nth-order differential equation (3) on an interval I is said to be a **fundamental set of solutions** on the interval.

DEFINITION 4.4 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (3) on an interval I. The **general solution** of the equation on the interval is defined to be

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Exam 2

4.1

4.3

4.2

4.4

DEFINITION 4.5 **General Solution—Nonhomogeneous Equations**

Let y_p be a given solution of the nonhomogeneous linear nth-order differential equation (4) on an interval I, and let

$$y_c = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

denote the general solution of the associated homogeneous equation (3) on the interval. The **general solution** of the nonhomogeneous equation on the interval is defined to be

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x).$$

Complementary Function In Definition 4.5 the linear combination

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

which is the general solution of (3), is called the **complementary function** for equation (4). In other words, the general solution of a nonhomogeneous linear differential equation is

$$y = \underbrace{\overset{y_c}{\textit{complementary function}}}_{\text{solution of corresponding homogeneous eq.}} + \underbrace{\overset{y_p}{\textit{any particular solution}}}_{\text{solution for a nonhomogeneous eq.}}.$$

any one should work

① to determine when a set of solutions is linearly indep / dep.

② to find general solution for homogeneous eq. (y_c)

③ to find particular solution for non-homogeneous e.q. (y_p)

a) Determine whether the given function are linearly indep/dep. on $(-\infty, \infty)$

$f_1(x) = x$ $\quad f_2(x) = x^2$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$w(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = \underset{a}{x} \cdot \underset{d}{2x} - \underset{c}{1} \cdot \underset{b}{x^2} = 2x^2 - x^2 = x^2$$

$x = 0$ $\quad w(x, x^2) = 0$

$x \neq 0$ $\quad w(x, x^2) \neq 0$ $\quad$ (for all $(-\infty, \infty) \setminus \{0\}$

$\Rightarrow x, x^2$ are linearly independent.

b) Determine whether the given function are linearly indep / dep. on $(-\infty, \infty)$

$f_1(x) = e^x$ $\qquad$ $f_2(x) = e^{-x}$

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^{-x} e^x$$

$$= -e^{\cancel{x}-x} - e^{x-x}$$

$$= -e^0 - e^0$$

$$= -1-1$$

$$= -2 \neq 0 \quad \text{for all } x \in (-\infty, \infty)$$

$\therefore$ e^x and e^{-x} are linearly independent.

c) Determine whether the given function are linearly indep/dep. on $(-\infty, \infty)$

$f_1(x) = \cos 2x \qquad f_2(x) = \cos^2 x \qquad f_3(x) = \sin^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$

$f_1 \quad f_2 \quad f_3$ are linearly dependent

$$\begin{array}{ccc} f_1 & f_2 & f_3 \\ \shortparallel & \shortparallel & \shortparallel \\ \cos 2x & \cos^2 x & \sin^2 x \end{array}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$$

$f_1(x) = x \qquad f_2(x) = x^{-2} \qquad f_3(x) = x^{-2}\ln x.$

$$W(x, x^{-2}, x^{-2}\ln x) = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & -2x^{-3}\ln x + x^{-3} \\ 0 & 6x^{-4} & 6x^{-4}\ln x - 2x^{-4} - 3x^{-4} \end{vmatrix} = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & (1-2\ln x)x^{-3} \\ 0 & 6x^{-4} & (6\ln x - 5)x^{-4} \end{vmatrix}$$

$$= x^{1-3-4}(-2)(6\ln x - 5) + 1\cdot 6x^{-4-2}\ln x - x^{-2-4}(6\ln x - 5) - 6x^{1-4-3}(1-2\ln x)$$

$$= (\underline{-12\ln x} + 10 + \underline{6\ln x} - \underline{6\ln x} + 5 - 6 + \underline{12\ln x})\,x^{-6} = 9x^{-6} \neq 0.$$

$\Rightarrow$ linearly independent.

$$y'' - 7y' + 10y = 24e^{x}$$

$$y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}, \quad (-\infty, +\infty)$$ general solution of nonhomogeneous

$y_1' = 2e^{2x}$ $y_1'' = 4e^{2x}$

$4e^{2x} - 7 \cdot 2e^{2x} + 10e^{2x} = 0.$

$y_2' = 5e^{5x}$ $y_2'' = 25e^{5x}$

$25e^{5x} - 35e^{5x} + 10e^{5x} = 0.$

$$\begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix} = 5e^{2x+5x} - 2e^{2x+5x}$$

$$= 3e^{7x} \neq 0.$$

$\Rightarrow$ $y_1 = e^{2x}$ and $y_2 = e^{5x}$ form a fundamental solution of homogeneous equation

$y' = 6e^{x}$ $y'' = 6e^{x}$

$6e^{x} - 7 \cdot 6e^{x} + 10 \cdot 6e^{x} = (60 - 42 + 6)e^{x} = 24e^{x}$

$\Rightarrow$ $y_p = 6e^{x}$ is a particular solution of the nonhomogeneous equation.

HW 4.1

3. 33. 35. 37. 39. 41. 43.

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Dx^2 + Ex + F)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + D)e^{3x}\sin 4x$

4.2. Finding 2nd Solution: for 2nd-order Linear differential equation

Given D.E. in a standard form $y''+P(x)y'+Q(x)y=0$ and $y_1(x)$ is one of the solutions. Then the 2nd solution is

$$y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx$$

* There is a way to get $y_2(x)$ without memorizing the form. if you are interested, then either read the textbook or ask me during the office hour :).

e.g.) $y''+5y'=0$ and $y_1(x)=1$.

$P(x)=5 \quad Q(x)=0$

$$y_2(x)=1\cdot\int\frac{e^{-\int 5dx}}{1^2}dx=\int e^{-5x}dx$$

$$=-\frac{1}{5}e^{-5x}$$

2nd solution: e^{-5x} ← we drop $-\frac{1}{5}$ since we write $\underline{c_2}e^{-5x}$ (general soln: $c_1+c_2e^{-5x}$)

why can we drop c? they only contribute constant terms of the solution. we can write them as $c\cdot y_2(x)$ in the general solution

What happens if you don't drop the constants?

$$y_2(x) = 1 \cdot \int \frac{e^{-\int 5\,dx}}{1^2}\,dx = \int e^{-5x+C}\,dx = -\frac{1}{5}\left(e^{-5x}e^{C}\right) + D$$

$$= -\frac{1}{5}e^{C}\underbrace{e^{-5x}}_{\text{2nd solution}} + D \cdot \underbrace{1}_{\text{1st solution}}$$

C D are constant.

So 2nd soln $y_2(x)$ is really e^{-5x} and the 1st soln $y_1(x)$ is 1.

The general soln $y(x) = c_1 e^{-5x} + c_2 \cdot 1 = c_1 e^{-5x} + c_2$.

$-\frac{1}{5}e^{C}$ would be a part of constant C_1.

* you can also use auxiliary equation $m^2 + 5m = 0$ to find the solution since the coefficients are constants.

$$m(m+5) = 0 \implies m = 0 \text{ or } m = -5$$

so general sol: $y(x) = c_1 e^{0x} + c_2 e^{-5x} = c_1 + c_2 e^{-5x}$.

e.g). $x^2y'' - 6y = 0$, $\underbrace{y_1(x) = x^3}_{\text{given}}$

* we can't use auxiliary equation since coefficients are not constants.

standard form: $y'' - \frac{6}{x^2}y = 0$

$P(x) = 0$ since there is no y' term.

$Q(x) = -\frac{6}{x^2}$

$$y_2(x) = x^3 \int \frac{e^{-\int 0\,dx}}{(x^3)^2}\,dx = x^3 \int \frac{e^{-c}}{x^6}\,dx$$ * c is a constant

$$= x^3 e^{-c} \int x^{-6}\,dx = x^3 e^{-c}\left[-\frac{1}{5}x^{-5}\right] = -\frac{1}{5}e^{-c}\underbrace{x^{-2}}_{\text{2nd}}.$$

2nd soln: x^{-2}.

general solution: $c_1x^3 + c_2x^{-2}$.

explain xe^{mx}

$$ay'' + by' + cy = 0.$$

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0.$$

$P(x)$

$$am^2 + bm + c = 0.$$

$$m^2 + \frac{b}{a}m + \frac{c}{a} = 0.$$

$$m_1 = m_2 = \frac{-\frac{b}{a} \pm \sqrt{\cancel{\left(\frac{b^2}{a}\right) - 4\frac{b}{a}\frac{c}{a}}}}{2}. \quad = 0$$

$$= -b/2a.$$

$$\boxed{e^{m_1 x}} \qquad x e^{m_1 x}.$$

$$\boxed{-b/a = 2m_1} = 2m_2.$$

$$y_1 = e^{m_1 x}$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int b/a\, dx}}{y_1^2}\, dx$$

$$= e^{m_1 x} \int \frac{e^{-\int b/a\, dx}}{e^{2m_1 x}}\, dx$$

$$= e^{m_1 x} \int \frac{e^{\int 2m_1 dx}}{e^{2m_1 x}}\, dx = e^{m_1 x} \int \frac{\cancel{e^{2m_1 x}}}{\cancel{e^{2m_1 x}}}\, dx = e^{m_1 x}\int 1 dx$$

$$= e^{m_1 x} \cdot x$$

eg.) (14) $4x^2y'' + y = 0$ $\quad \underbrace{y_1 = x^{\frac{1}{2}}\ln x}_{\text{given.}}$

$P(x) = 0.$ $\quad y'' + \frac{1}{4x^2}y = 0.$

we have $\quad y_2 = x^{\frac{1}{2}}\ln x \int \frac{e^{-\int 0 dx}}{x(\ln x)^2}dx$

$= x^{\frac{1}{2}}\ln x \left(-\frac{1}{\ln x}\right)e^{c} = -x^{\frac{1}{2}}e^{c}.$

$y_2 = x^{\frac{1}{2}}.$

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eg.) (14a) $4x^2y'' + y = 0$ $\quad y_1 = x^{\frac{1}{2}}$

$P(x)$, $\quad y'' + \frac{1}{4x^2}y = 0$

we have $\quad y_2 = x^{\frac{1}{2}}\int \frac{e^{-\int 0 dx}}{x}dx$

$= x^{\frac{1}{2}}\ln x$

$y_2 = x^{\frac{1}{2}}\ln x.$

HW 4.2:

1. 3. 5. 7. 9. 11. 13. 15.

4.3. Homogeneous Linear eq with constant coefficients.

Consider a linear 2nd order homogeneous D.E.

$$ay'' + by' + cy = 0 \qquad (1).$$

a.b.c. are real number.

and 2nd order equation $am^2 + bm + cm^0 = 0 \qquad (2).$

equation (2) is called the auxiliary eq. of D.E. (1)

Solution of (2) will give us solution of (1)

Case 1: m_1 and m_2 are distinct real roots. (means $a(m_1)^2 + b(m_1) + c = 0$, $a(m_2)^2 + b(m_2) + c = 0$)

in this case, $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$

are solutions of (1). Thus general solution of (1) on $(-\infty, \infty)$

is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

e.g.) $y'' - y' - 6y = 0$ a.eq. $m^2 - m - 6 = (m-3)(m+2) = 0.$

solution : $m = 3, -2.$

$y = c_1 e^{3x} + c_2 e^{-2x}$

(c_1 and c_2 can be solved with initial value).

Case 2: $m_1 = m_2$ and real.

in this case, $y_1 = c_1 e^{m_1 x}$ and $y_2 = c_2 x e^{m_1 x}$ are solution of (1). Thus general solution of (1) on $(-\infty, \infty)$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

e.g) $y'' + 4y' + 4y = 0$

a.eq. $m^2 + 4m + 4 = (m+2)(m+2) = 0$.

solution: $m = -2$ (repeated root)

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Case 3. m_1 and m_2 are complex and conjugates of each other.

i.e. $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ (it can be $m_1 = i\beta$, $m_2 = -i\beta$)

in this case, $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ is the general solution of (1). If $m_1 = i\beta$ and $m_2 = -i\beta$ $(\alpha = 0)$, then $y = c_1 \cos \beta x + c_2 \sin \beta x$. is the general solution.

e.g.) $y'' + y' + 2y = 0$

a. eq. $m^2 + m + 2 = 0$.

$$m = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{-7}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \qquad \alpha = \frac{1}{2} \quad \beta = \frac{\sqrt{7}}{2}$$

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{7}}{2}x + c_2 \sin \frac{\sqrt{7}}{2}x\right)$$

e.g.) $y'' + 9y = 0$

a. eq. $m^2 + 9 = 0$

$m^2 = -9$

$m = \pm\sqrt{-9} = \pm 3i$, $\alpha = 0$ $\beta = 3$.

$$y = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$= c_1 \cos 3x + c_2 \sin 3x$$

Higher order D.E.

Same idea works. Say $(m-a)(m-b)(m-c)=0$ is a.eq of some DE.
where a, b and c are real numbers and all distinct.
then $y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx}$ will be the general sol of the D.E.

If $(m-a)^3=0$ is a.eq. of some D.E. then

$$y = c_1 e^{ax} + c_2 x e^{ax} + c_3 x^2 e^{ax}$$

is the general sol of the D.E.

eg) Solve $y'''' + 2y'' + y = 0$ a.eq. $m^4 + 2m^2 + 1 = 0$.

$(m^2+1)(m^2+1)=0$.

$m^2=-1 \quad m^2=-1$

$m = \pm i \quad m = \pm i \quad (\alpha=0, \beta=\pm 1)$

so $\pm i$ is a repeated root.

Since roots are repeated, we get

$$y = c_1 e^{ix} + c_2 e^{-ix} + c_3 x e^{ix} + c_4 x e^{-ix}$$

$$\overset{*}{=} c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$

(relabeling)

* Euler formula: $e^{ix} = \cos x + i \sin x$, $e^{-ix} = \cos x - i \sin x$.

Higher order D.E

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0.$$

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0 = 0.$$

$$(m - \underline{m_1})^n = 0. \qquad m_1 \cdot n \text{ times.}$$

$$e^{m_1 x}, \ x e^{m_1 x}, \ x^2 e^{m_1 x}, \ \cdots\cdots, \ x^{n-1} e^{m_1 x} \text{ — Linearly indep}$$

general soln :

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \cdots + c_n x^{n-1} e^{m_1 x}.$$

$\boxed{20}$ $4y''' + 4y'' + y' = 0.$

$$4m^3 + 4m^2 + m = 0.$$

$$m(4m^2 + 4m + 1) = 0$$

$$m(2m+1)^2 = 0$$

$$m_1 = 0 \quad m_2 = -\tfrac{1}{2} \quad m_3 = -\tfrac{1}{2}.$$

$$y = c_1 e^{0x} + c_2 \underline{e^{-\frac{1}{2}x}} + c_3 \underline{x e^{-\frac{1}{2}x}}$$

$\boxed{18}$ $2y'' + 2y' + y = 0$

$2m^2 + 2m + 1 = 0.$

$$m = \frac{-2 \pm \sqrt{2^2 - 8}}{2 \cdot 2}$$

$$= \frac{-2 \pm 2\sqrt{2}\,i}{2 \cdot 2}$$

$$= \frac{-1 \pm \sqrt{2}\,i}{2}.$$

$\alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{2}}{2}.$

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \frac{\sqrt{2}}{2}x\right).$$

4.4. Undetermined Coefficients. – Superposition approach.

Q: When can we use "undetermined coefficients"?

A: D.E. of type $ay'' + by' + cy = g(x)$ where a, b, c are real numbers (constants) and $g(x) =$
- constant
- polynomial
- $\sin\beta x$, $\cos\beta x$
- $e^{\alpha x}$
- finite sum and product of

* if $g(x) = 0$, then $ay'' + by' + cy = 0$ is a homogeneous, linear, degree 2 D.E. So we can use auxiliary eq to solve

So. $g(x) \neq 0$.

* higher order can also be solved as long as coeff:s are constants.

** If $g(x) \neq 0$, then $ay'' + by' + cy = g(x)$ is a non-homogeneous - linear degree 2 D.E. We first find a general solu for $ay'' + by' + cy = 0$. which will be denoted as y_c (complementary function) and we find any one solution that works for $ay'' + by' + cy = g(x)$. denoted as y_p (particular solution).

e.g.① Solve $y''+2y'-3y = 4x+2$.

step 1: Find $y_c(x)$ solve $y''+2y'-3y=0$. a.eq $m^2+2m-3=0$

$(m+3)(m-1)=0$

$m=-3,\ m=1$.

$$y_c(x) = c_1 e^{-3x} + c_2 e^{x}$$

step 2: Find $y_p(x)$

since $y''+2y'-3y = 4x+2$, y has to be some kind of polynomial

So we guess that $y_p = Ax+B$, A, B a real number.

Since y_p is a solution of $y''+2y'-3y = 4x+2$. we should have

$$y_p''+2y_p'-3y_p = 4x+2.$$

$\boxed{y_p = -\frac{4}{3}x - \frac{14}{9}}$

$y_p(x) = Ax+B$ $\quad y_p'(x) = A$ $\quad y_p''(x)=0$.

so $y_p''+2y_p'-3y_p = 0 + 2(A) - 3(Ax+B) = -3Ax-3B+2A$

Should be $= 4x+2$.

so $-3Ax = 4x$ and $-3B+2A=2$ $\Rightarrow A = -\frac{4}{3}$ $\quad -3B+(-\frac{4}{3})\cdot 2 = 2$

$\Rightarrow B = -\frac{1}{3}(\frac{14}{3}) = -\frac{14}{9}$

Step 3. Find the general solution

y (general soln) $= y_c + y_p = c_1 e^{-3x} + c_2 e^{x} - \frac{4}{3}x - \frac{14}{9}$.

* There is a lot of algebra used to find $y_p(x)$.
prepare to spend some time with homework.

** Guessing the correct form of $y_p(x)$ is very important in this solution. try to guess the correct form of $y_p(x)$ with HW.

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Dx^2 + Ex + F)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + D)e^{3x}\sin 4x$

use this table to guess the form of $y_p(x)$

example ②

$y'' - y' + y = 2\sin 3x$

$m^2 - m + 1 = 0$

$m = \dfrac{1 \pm \sqrt{(-1)^2 - 4}}{2}$.

$m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

$\alpha = \frac{1}{2} \quad \beta = \frac{\sqrt{3}}{2}$.

$y_c = e^{\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x + \sin\frac{\sqrt{3}}{2}x\right)$.

example.

$y_p = A\sin 3x + B\cos 3x$.

$y_p' = 3A\cos 3x - 3B\sin 3x$

$y_p'' = -9A\sin 3x - 9B\cos 3x$

$\underbrace{-9A\sin 3x - 9B\cos 3x}_{y_p''} \underbrace{- 3A\cos 3x + 3B\sin 3x}_{y_p'} + \underbrace{A\sin 3x + B\cos 3x}_{y_p} = 2\sin 3x$.

$(-9A + 3B + A)\sin 3x + (-9B - 3A + B)\cos 3x = 2\sin 3x + 0$

$-8A + 3B = 2$.

$-8B - 3A = 0$.

$-64A + 24B = 16$

$-24B - 9A = 0$.

$-64A - 9A = 16$.

$-73A = 16$.

$A = -\frac{16}{73}$.

$B = -\frac{3}{8}A = -\frac{3}{8} \cdot -\frac{\cancel{16}\,2}{73} = \frac{6}{73}$.

$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$.

example 3

$$y'' - y' + y = e^{5x}$$

HW. 4.4.

1. 3 5. 7. 9. 11. 13, 15

$y_p = ?$ $\quad Ae^{5x}$.

$y_p' = 5Ae^{5x}$

$y_p'' = 25Ae^{5x}$.

$25Ae^{5x} - 5Ae^{5x} + Ae^{5x} = e^{5x}$

$(25A - 5A + A)\, e^{5x} = e^{5x}$

$21A = 1$.

$A = \frac{1}{21}$.

$y_p = \frac{1}{21} e^{5x}$.

When the table does not work.

ex) 4 $y''+2y'-3y=e^x$

Step 1: Find $y_c(x)$

a. eq. $m^2+2m-3=0$

$(m+3)(m-1)=0$

$m=-3, 1$

$y_c(x)=c_1e^{-3x}+c_2e^x$

Step 2. Find $y_p(x)$

$g(x)=e^x$ so $y_p(x)=Ae^x$ (used table).

then $y_p'(x)=Ae^x$ and $y_p''(x)=Ae^x$

so $y_p''+2y_p'-3y_p=Ae^x+Ae^x-3Ae^x$

$=0$

$=e^x$ (since $y''+2y'-3y=e^x$).

which means $0=e^x$ ... which is not possible.

So what went wrong?

Solution $y_c(x)=c_1e^{-3x}+\underline{c_2e^x}$

$y_c(x)$ is not linearly independent with $y_p(x)=Ae^x$

we need to have y_p and y_c to be linearly independent

in that case, Let $y_p(x)=xAe^x$

, multiply x to your guess of $y_p(x)$

Step 2 (take 2): Find $y_p(x)$

$g(x) = e^x$ so $y_p(x) = Ae^x$ but since $y_c(x) = C_1 e^{-3x} + c_2 e^x$

Let $y_p(x) = xAe^x$.

$y_p' = Ae^x + Axe^x$

$$y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x.$$

$$y_p'' + 2y_p' - 3y_p = (2Ae^x + Axe^x) + 2(Ae^x + Axe^x) - 3Axe^x$$

$$= 2Ae^x + 2Ae^x + \underbrace{Axe^x + 2Axe^x - 3Axe^x}_{} = 0$$

$$= 4Ae^x$$

should be $= e^x$

so $4Ae^x = e^x$

$4A = 1$ and $A = \frac{1}{4}$. So $y_p(x) = \frac{1}{4} x e^x$

<u>Step 3</u>: Find the general solution

$$y = C_1 e^{-3x} + C_2 e^x + \frac{1}{4} x e^x.$$

More example on choosing the right y_p.

e.g.) 5 $y''+3y'+2y=6$

m^2+3m+2

$(m+2)(m+1)$

$m=-2,-1.$

$y_c(x)=c_1e^{-2x}+c_2e^{-x}$

$y_p(x)=A.$

e.g.) 6 $y''+3y=-48x^2e^{3x}$

$m^2+3=0$

$m=\pm\sqrt{3}i$

$y_c(x)=c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x$

$y_p(x)=(Ax^2+Bx+c)e^{3x}$

e.g.) 7 $y''+4y=3\sin 2x$

$(m^2+4=0)$

$m=\pm 2i$

$y_c(x)=c_1\cos 2x+c_2\sin 2x$

$y_p(x)=A\cos 2x+B\sin 2x$

which contains $y_c(x)$ so multiply x to y_p.

use $y_p(x)=Ax\cos 2x+Bx\sin 2x$

eg.) ⑨ $y'' + y = 2x \sin x$ $\quad (m^2+1=0)$

$m = \pm i.$

$y_c(x) = c_1 \cos x + c_2 \sin x$

$y_p(x) = (Ax+B)\cos x + (Cx+D)\sin x$

But $B\cos x + D\sin x$ is part of $y_c(x)$.

So multiply x to $y_p(x)$

use $y_p(t) = (Ax^2+Bx)\cos x + (Cx^2+Dx)\sin x.$

e.g.) ⑩ $y'' - 2y' + 5y = e^x \cos 2x$ $\quad (m^2 - 2m + 5)$

$$m = \frac{2 \pm \sqrt{4-4(1)(5)}}{2.} = \frac{2 \pm \sqrt{-16}}{2.}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$y_c(x) = e^x(c_1 \cos 2x + c_2 \sin 2x)$

$y_p(x) = Ae^x \cos 2x + Be^x \sin 2x$

But $y_p(t)$ canbe obtained from $y_c(x)$

So, multiply x to $y_p(x)$

* $y_p(x) = Axe^x \cos 2x + Bxe^x \sin 2x$

e.g.) 11 $y'' + 2y' - 24y = 16 - (x+2)e^{4x}$

$m^2 + 2m - 24 = 0$

$(m+6)(m-4) = 0$

$m = 4, 6$

$y_c(x) = c_1 e^{4x} + c_2 e^{-6x}$

$$y_p(x) = \underbrace{A}_{y_{p_1}} + \underbrace{(Cx+D)e^{4x}}_{y_{p_2}}.$$

De^{4x} in y_{p_2} can be obtained from $c_1 e^{4x}$

multiply x to y_{p_2} not all of y_p (but just $y_{p_2} = (Cx+D)e^{4x}$)

use $y_p(x) = A + (Cx^2 + Dx)e^{4x}$.

eg. (12)): $y'''' - 2y''' + y'' = 0$

$m^4 - m^2 + 2m = 0.$

$m^2(m^2 - 2m + 1) = 0$

$m^2(m-1)(m-1) = 0$

$m = 0,\ 1.$ repeated.

$$y = y_c = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{x} + c_4 x e^{x}$$

$$= c_1 + c_2 x + c_3 e^{x} + c_4 x e^{x}$$

eg. (13)): $y'' + 2y' - 3y = e^{2x} + x$,

$m^2 + 2m - 3 = 0.$

$(m+3)(m-1) = 0$

$m = -3,\ 1.$

$$y_c(x) = c_1 e^{x} + c_2 e^{-3x}$$

$$y_p(x) = \underbrace{Ae^{2x}}_{y_{p_1}} + \underbrace{Bx + C}_{y_{p_2}}$$

$$y_p'(x) = 2Ae^{2x} + B$$

$$y_p''(x) = 4Ae^{2x}$$

$$y_p'' + 2y_p' - 3y_p = 4Ae^{2x} + 2(2Ae^{2x} + B) - 3(Ae^{2x} + Bx + C)$$

$$= 4Ae^{2x} + 4Ae^{2x} - 3Ae^{2x} - 3Bx + 2B - BC.$$

$$= 5Ae^{2x} - 3Bx + 2B - 3C. = e^{2x} + x$$

$A = \frac{1}{5}$ $B = -\frac{1}{3}$ $C = \frac{2}{9}$ $y_p(x) = \frac{1}{5}e^{2x} - \frac{1}{3}x + \frac{2}{9}$

$y = c_1 e^{x} + c_2 e^{-3x} + \frac{1}{5}e^{2x} - \frac{1}{3}x + \frac{2}{9}$

eg. 14) $y'' + 2y' + 2y = e^{-x}\sin(x)$

$m^2 + 2m + 2 = 0$ $\quad m = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$

$\alpha = -1 \quad \beta = 1$

$y_c(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$.

$y_p(x) = Ae^{-x}\sin x + Be^{-x}\cos x$ Nope!

$= Axe^{-x}\sin x + Bxe^{-x}\cos x$.

eg. 15) $y'' + y' - 2y = e^x + xe^{2x}$

$m^2 + m - 2 = 0 \quad (m+2)(m-1) = 0 \quad -2. \quad 1.$

$y_c(x) = c_1 e^{-2x} + c_2 e^{x}$.

$y_p(x) = \underbrace{Ae^{x}}_{y_{p_1}} + \underbrace{(Bx + C)e^{2x}}_{y_{p_2}}$ Nope!

$y_p(x) = xAe^{x} + (Bx + C)e^{2x}$.

4.6 Variation of Parameters.

Used for 2nd order linear D.E $y'' + P(x)y' + Q(x)y = f(x)$

(So any problem from 4.4 undetermined coeff can be used here as well). this method will be simpler & faster.

Step1 : Complementary function $y_c = c_1 y_1 + c_2 y_2$. (since D.E. is deg 2 y_c has two parts)

Step 2 : Compute $W = W(y_1, y_2)$ ← Wronskian.

$$= \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step3 : compute $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

Step4 : compute $u_1' = \frac{W_1}{W}$ and $u_2' = \frac{W_2}{W}$

Step5 : compute $u_1 = \int u_1' \, dx$ and $u_2 = \int u_2' \, dx$

Step 6 : particular solution $y_p = u_1 y_1 + u_2 y_2$

step 7 : General solution $y = y_p + y_c$

* There are lots of steps.! Make sure you are careful with W_1 and W_2.

** Not all non-homogeneous D.E. can be solved using undet coeff.

$g(x)$ needs to be of certain type with variation parameters

$g(x)$ can be any type of function.

e.g.) $y'' + y = \sec x$

step 1. Find $y_c(x)$

a. eq. $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i \quad \alpha = 0, \beta = 1$

$$y_c(x) = e^{0x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$$

$y_1 = \cos x \quad y_2 = \sin x$

* it does not matter which one you should choose as y_1 and y_2. if we let $y_1 = \sin x$ and $y_2 = \cos x$. we will still get the same answer.

step 2. compute $W = W(y_1, y_2)$

$$W = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \det \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (\sin x)^2$$
$$= \cos^2 x + \sin^2 x = 1.$$

step 3. compute W_1 and W_2.

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = 0 \cdot \cos x - \sin x \sec x = -\frac{\sin x}{\cos x} = -\tan x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x - 0 \cdot (-\sin x) = \frac{\cos x}{\cos x} = 1.$$

step 4. compute u_1' and u_2'

$$u_1' = \frac{W_1}{W} = \frac{-\tan x}{1} = -\tan x \qquad u_2' = \frac{W_2}{W} = \frac{1}{1} = 1.$$

step 5. compute u_1 and u_2.

$u = \cos x$

$du = -\sin x\, dx$

$$u_1 = \int u_1' dx = \int -\tan x\, dx = \int -\frac{\sin x}{\cos x} dx = \int \frac{1}{u} du = \ln|\cos x|$$

$$u_2 = \int u_2' dx = \int 1\, dx = x.$$

* constant will only contribute another constant in the soln so we can drop it in this step :)

Step 6. Particular solution.

$$y_p(x) = u_1 y_1 + u_2 y_2 = \ln|\cos x| \cos x + x \sin x$$

Step 7. General solution

$$y(x) = c_1 \cos x + c_2 \sin x + \ln|\cos x| \cos x + x \sin x.$$

$y(x)$ is the solution to the given D.E. for any x in the interval not including $\frac{\pi n}{2}$, n odd, since $\cos(\frac{\pi n}{2}) = 0$ and $\ln(\cos\frac{\pi n}{2}) =$ undefined.

Somtimes $\int u_1' dx$ and $\int u_2' dx$ can not be expressed in terms of elementary functions. Then we express them in integral form.

e.g.) $y'' - 4y = \frac{e^{2x}}{x}$

Step 1: Find $y_c(x)$

a.eq. $m^2 - 4 = 0$

$m^2 = 4$, $m = \pm 2$.

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$y_1 = e^{2x}$ $\quad y_2 = e^{-2x}$

Step 2: Compute $W = W(y_1, y_2)$

$$W = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = e^{2x}(-2e^{-2x}) - 2e^{2x}(e^{-2x})$$

$$= -2e^{2x+(-2x)} - 2e^{2x+(-2x)}$$

$$= -2e^0 - 2e^0 = -2-2 = -4.$$

Step 3: Compute W_1 and W_2

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = 0 - e^{-2x}\left(\frac{e^{2x}}{x}\right) = -\frac{1}{x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = e^{2x}\left(\frac{e^{2x}}{x}\right) - 0 = \frac{e^{4x}}{x}.$$

Step 4: $u_1' = \frac{W_1}{W} = \frac{-\frac{1}{x}}{-4} = \frac{1}{4x}$ $\qquad u_2' = \frac{\frac{e^{4x}}{x}}{-4} = \frac{e^{4x}}{-4x}$

Step 5: Compute u_1 and u_2.

$$u_1 = \int u_1' dx = \int \frac{1}{4x} dx = \frac{1}{4}\ln x$$

$$u_2 = \int u_2' dx = \int \frac{e^{4x}}{-4x} dx \Leftarrow$$

this is the problem integral can not be expressed. interms of elt. functions.

so let $u_2 = \int_{x_0}^{x} \frac{e^{4t}}{-4t} dt$.

Step 6: Particular solution

$$y_p(x) = u_1 y_1 + u_2 y_2 = e^{2x} \frac{1}{4} \ln x + e^{-2x} \int_{x_0}^{x} \frac{e^{4t}}{-4t} dt.$$

Step 7: general solution.

$$y = y_c + y_p = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4} e^{2x} \ln x + e^{-2x} \int_{x_0}^{x} \frac{e^{4t}}{-4t} dt$$

4.7 Cauchy-Euler Equation.

Any linear differential equation of the form

$$\underbrace{a_n x^n y^{(n)}}_{\text{same}} + a_{n-1} x^{n-1} \underbrace{y^{(n-1)}}_{\text{same.}} + \cdots + a_1 x y' + a_0 y = g(x),$$

where $a_n, a_{n-1}, \cdots, a_0$ are constants.

Cauchy-Euler equation.

* degree of each monomial coefficient = order of differentiation.

Consider D.E. of the form $ax^2y'' + bxy' + cy = 0$

We try solution $y = x^m$

Then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$

Substitute y'' and y' into the equation.

$$ax^2 \underbrace{m(m-1)x^{m-2}}_{y''} + bx \underbrace{m x^{m-1}}_{y'} + \underbrace{cx^m}_{y} = 0$$

$$ax^2 m(m-1)x^m + bmx^m + cx^m = 0.$$

(Remember 4.3)

$$x^m(am(m-1) + bm + c) = 0.$$

$$x^m(\underbrace{am^2 + (b-a)m + c}) = 0 \rightarrow \text{2n deg polynomial. this is the auxiliary equation}$$

Since we have deg 2 polynomial, we have 2 root, say m_1 and m_2.

Case 1: $m_1 \neq m_2$, m_1 and m_2 are real numbers.

In this case, soln to the correspondig P.E. is

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

example). Solve $x^2 y'' - xy' = 0$.

$y = x^m \quad y' = m x^{m-1} \quad y'' = m(m-1) x^{m-2}$

$x^2 m(m-1) x^{m-2} - x m x^{m-1} = 0$

$m(m-1)x^m - m x^m = 0$

$x^m (\underbrace{m^2 - m - m}) = 0$

a. eq. $\quad m(m-2) = 0 \quad m = 0$ or $m = 2$.

general soln: $C_1 X^0 + C_2 X^2 = C_1 + C_2 X^2$.

Case 2 : $m_1 = m_2$ (let say m_1 is a real number).

In this case, soln to the corresponding d.e is $y = c_1 x^{m_1} + c_2 x^{m_1} \underbrace{\ln x}$

extra term attached.

example) Solve $4x^2 y'' + y = 0$

$y = x^m \qquad y' = m x^{m-1} \qquad y'' = m(m-1) x^{m-2}$

$4x^2 m(m-1) x^{m-2} + x^m = 0$

$4m(m-1) x^m + x^m = 0$

$x^m (4m^2 - 4m + 1) = 0$

a.eq. $4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0$

$m = \frac{1}{2}$ repeated.

general soln : $y = c_1 x^{\frac{1}{2}} + c_2 x^{\frac{1}{2}} \ln x$

Case 3: m_1 and m_2 are conjugate complex numbers i.e. $\alpha \pm \beta i$ are soln

In this case, soln to the corresponding D.E. is

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$$

example) $x^2 y'' + 3xy' + 3y = 0.$

$$x^2 m(m-1)x^{m-2} + 3x\, m x^{m-1} + 3x^m = 0.$$

$$m(m-1)x^m + 3m x^m + 3x^m = 0.$$

$$x^m\left(m(m-1) + 3m + 3\right) = 0.$$

$$x^m\left(m^2 - m + 3m + 3\right) = 0$$

$$x^m\left(\underline{m^2 + 2m + 3}\right) = 0.$$

$$m = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = \underbrace{-1}_{\alpha} \pm \underbrace{\sqrt{2}}_{\beta} i$$

general soln is

$$y = x^{-1}\left[c_1 \cos(\sqrt{2} \ln x) + c_2 \sin(\sqrt{2} \ln x)\right]$$

6.1 is similar + different from 4.3.

Q. Can we solve non-homogeneous eq. too?

An: Yes.

1. get complementary soln $y_c(x)$ using above methods.
2. get particular soln using variation of parameters (4.7) or undetermined coefficients method.

Then the general solution is $y_c + y_p$ (Done! :))

Find a general solution to the Cauchy-Euler equation.

$$x^2y''-3xy'+3y=2x^4e^x$$

---

try $y=x^m$. $x^2m(m-1)x^{m-2}-3xmx^{m-1}+3x^m=0$

$$(m(m-1)-3m+3)x^m=0.$$

$m^2-4m+3=0$ $(m-1)(m-3)=0$ $m_1=1$ or $m_2=3$.

So that $\{\underset{y_1}{x}, \underset{y_2}{x^3}\}$ is a fundamental solution set for the correspondly homogeneous equation. Next variation of parameters method. $\boxed{y''-3x^{-1}y'+3x^{-2}y=2x^2e^x}$

Step1: Complementary function $y_c=C_1y_1+C_2y_2 = C_1x+C_2x^3$ (since deg 2).

Step2: Compute $W=W(y_1,y_2)=\det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\det\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}=3x^3-x^3=2x^3$

Step3: Compute $W_1=\det\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix}=-2x^5e^x$. $W_2=\det\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}=2x^3e^x$

$f(x)=2x^2e^x$

Step 4: compute $u_1'=\frac{W_1}{W}=\frac{-2x^5e^x}{2x^3}=-x^2e^x$

and $u_2'=\frac{W_2}{W}=\frac{2x^3e^x}{2x^3}=e^x$

Step 5: compute $u_1 = \int u_1' \, dx = \int -x^2 e^x \, dx = -x^2 e^x + \int e^x 2x \, dx$

$= -x^2 e^x + 2e^x x - \int 2 e^x dx.$

$= -x^2 e^x + 2e^x x - 2e^x$

$u_2 = \int u_2' \, dx = \int e^x dx = e^x.$

Step 6: particular solution $y_p = u_1 y_1 + u_2 y_2$

$= (-x^2 e^x + 2x e^x - 2e^x) x + e^x x^3$

$= \cancel{-x^3 e^x} + 2x^2 e^x - 2x e^x + \cancel{e^x x^3}$

$= 2x^2 e^x - 2x e^x$

Step 7: general solution $y = y_c + y_p$

$= c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x$

e.g ①) $4x^2y'' + y = 0$.

$y = x^m$ $\quad 4x^2 m(m-1)x^{m-2} + x^m = 0$.

$x^m(4m(m-1)+1) = 0$

$4m^2 - 4m + 1 = 0$.

$(2m-1)^2 = 0$.

$m_1 = \frac{1}{2}$ or $m_2 = \frac{1}{2}$. (deg 2, $\therefore$ 2 solution).

$y = c_1 x^{\frac{1}{2}} + c_2 x^{\frac{1}{2}} \underbrace{\ln x}_{\text{attached extra term.}}$.

e.g ②). $xy'' - y' = 0$.

try $y = x^m$. $\quad x\,m(m-1)x^{m-2} - m x^{m-1} = 0$.

$x^m(m(m-1) - m) = 0$.

$m^2 - 2m = 0$.

$m(m-2) = 0$

$m_1 = 0$ or $m_1 = 2$. (deg < 2 so 2 solution form)

$y = c_1 x^0 + c_2 x^2 = c_1 + c_2 x^2$.

e.g 3) $x^2y''+5xy'+3y=0$

try $y=x^m$. $\quad x^2 m(m-1)x^{m-2}+5x\,m\,x^{m-1}+3x^m=0.$

$$x^m(m^2-m+5m+3)=0.$$

$$m^2+4m+3=0.$$

$$(m+1)(m+3)=0.$$

$m=-1$ or $m=-3$ (two real)

$$y=c_1x^{-1}+c_2x^{-3}$$

e.g (4) $4x^2y'' + 4xy' - y = 0.$

try $y = x^m$ $\quad 4x^2 m(m-1)x^{m-2} + 4xmx^{m-1} - x^m = 0.$

$x^m(4m(m-1) + 4m - 1) = 0.$

$4m^2 - 4m + 4m - 1 = 0$

$4m^2 = 1.$

$m^2 = \frac{1}{4}$

$m_1 = \frac{1}{2}$ or $m_2 = -\frac{1}{2}.$ (deg 2. so two solution).

(2 real)

general solution:

$y = c_1 x^{\frac{1}{2}} + c_2 x^{-\frac{1}{2}}$

e.g (5) $x^2y'' + 3xy' - 4y = 0$

try $y = x^m$

$$x^2 m(m-1)x^{m-2} + 3xmx^{m-1} - 4x^m = 0.$$

$$x^m(m^2 - m + 3m - 4) = 0.$$

$$m^2 + 2m - 4 = 0.$$

$$m = \frac{-2 \pm \sqrt{4+16}}{2} = \frac{-1 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}.$$

two real solution. $m_1 = -1 + \sqrt{5}$, $m_2 = -1 - \sqrt{5}$

$$y = c_1 x^{-1+\sqrt{5}} + c_2 x^{-1-\sqrt{5}}.$$

e.g. ⑥) $x^2y'' + 8xy' + 6y = 0$

try $y = x^m$.

$$x^2 m(m-1)x^{m-2} + 8x m x^{m-1} + 6x^m = 0.$$

$$x^m(m^2 - m + 8m + 6) = 0.$$

$$m^2 + 7m + 6 = 0.$$

$$(m+6)(m+1) = 0.$$

$m_1 = -6$ or $m_2 = -1$ (check 2. ⇒ 2 soln).

two real

$$y = c_1 x^{-6} + c_2 x^{-1}$$

eg. 7) $x^2y'' - 7xy' + 41y = 0$.

try $y = x^m$.

$$x^2 m(m-1)x^{m-2} - 7x\, m\, x^{m-1} + 41x^m = 0.$$

$$x^m(m^2 - m - 7m + 41) = 0.$$

$$m^2 - 8m + 41 = 0.$$

$$m = \frac{8 \pm \sqrt{64 - 41\cdot 4}}{2} = \frac{8 \pm \sqrt{64-164}}{2} = \frac{8 \pm 10i}{2}.$$

$$= 4 \pm 5i. \qquad \text{(complex conjugate)}.$$

$\alpha = 4$, $\beta = 5$.

$$y = x^4\left[c_1 \cos(5\ln x) + c_2 \sin(5\ln x)\right].$$

eg. 8) $2x^2y'' + xy' + y = 0.$

try $y = x^m$

$$2x^2 m(m-1)x^{m-2} + x\,m\,x^{m-1} + x^m = 0.$$

$$x^m(2m(m-1) + m + 1) = 0$$

$$2m^2 - 2m + m + 1 = 0$$

$$2m^2 - m + 1 = 0.$$

$$m = \frac{1 \pm \sqrt{1 - 4\cdot 2}}{4} = \frac{1 \pm \sqrt{-7}}{4} = \underbrace{\frac{1}{4}}_{\alpha} \pm \underbrace{\frac{\sqrt{7}}{4}}_{\beta}\, i.$$

(complex conjugate)

$$y = x^{\frac{1}{4}}\left[c_1 \cos\left(\frac{\sqrt{7}}{4}\ln x\right) + c_2 \sin\left(\frac{\sqrt{7}}{4}\ln x\right)\right].$$